

Sandwiches Missing Two Ingredients of Order Four

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Abstract

For a set \mathcal{F} of graphs, an instance of the \mathcal{F} -FREE SANDWICH PROBLEM is a pair (G_1, G_2) consisting of two graphs G_1 and G_2 with the same vertex set such that G_1 is a subgraph of G_2 , and the task is to determine an \mathcal{F} -free graph G containing G_1 and contained in G_2 , or to decide that such a graph does not exist. Initially motivated by the graph sandwich problem for trivially perfect graphs, which are the $\{P_4, C_4\}$ -free graphs, we study the complexity of the \mathcal{F} -FREE SANDWICH PROBLEM for sets \mathcal{F} containing two non-isomorphic graphs of order four. We show that if \mathcal{F} is one of the sets $\{\text{diamond}, K_4\}$, $\{\text{diamond}, C_4\}$, $\{\text{diamond}, \text{paw}\}$, $\{K_4, \overline{K_4}\}$, $\{P_4, C_4\}$, $\{P_4, \overline{\text{claw}}\}$, $\{P_4, \overline{\text{paw}}\}$, $\{P_4, \overline{\text{diamond}}\}$, $\{\text{paw}, C_4\}$, $\{\text{paw}, \text{claw}\}$, $\{\text{paw}, \overline{\text{claw}}\}$, $\{\text{paw}, \overline{\text{paw}}\}$, $\{C_4, \overline{C_4}\}$, $\{\text{claw}, \overline{\text{claw}}\}$, and $\{\text{claw}, \overline{C_4}\}$, then the \mathcal{F} -FREE SANDWICH PROBLEM can be solved in polynomial time, and, if \mathcal{F} is one of the sets $\{C_4, K_4\}$, $\{\text{paw}, K_4\}$, $\{\text{paw}, \overline{K_4}\}$, $\{\text{paw}, \overline{C_4}\}$, $\{\text{diamond}, \overline{C_4}\}$, $\{\text{paw}, \overline{\text{diamond}}\}$, and $\{\text{diamond}, \overline{\text{diamond}}\}$, then the decision version of the \mathcal{F} -FREE SANDWICH PROBLEM is NP-complete.

Keywords: Graph sandwich problem; forbidden induced subgraph

1 Introduction

Graph sandwich problems [12] are a natural generalization of recognition problems, and have received considerable attention [5, 7–10, 13, 18, 19]. It is not unusual that graph classes for which the recognition is very easy lead to challenging graph sandwich problems, which are either intractable or require interesting structural and algorithmic arguments for their solution. In such a situation, the graph sandwich problem motivates a detailed analysis of the corresponding graph class leading to insights that were probably not needed for some efficient ad-hoc recognition algorithm but are essential for the solution of the sandwich problem.

Good examples for this effect are graph classes defined by a finite set \mathcal{F} of forbidden induced subgraphs. In [7] Dantas, de Figueiredo, da Silva, and Teixeira initiated the study of graph sandwich problems for \mathcal{F} -free graphs, where \mathcal{F} contains a single graph. In [8] Dantas, de Figueiredo, Maffray, and Teixeira provided further results along this line, and, in particular, settled the complexity status of the graph sandwich problem for $\{F\}$ -free graphs for every graph F of order four. Considering forbidden induced subgraph of order four is rather natural, because many well known graph classes [3] are defined by one or more such graphs, and various aspects of these classes have been studied [2, 6, 14].

Originally motivated by the graph sandwich problem for trivially perfect graphs, which are the $\{P_4, C_4\}$ -free graphs, and following a suggestion by Golumbic, we initiate the study of the graph sandwich problem for \mathcal{F} -free graphs, where \mathcal{F} is a set of two non-isomorphic graphs of order four. In order to obtain our results, we rely on known results [1, 4, 15–17] for some cases, and develop new arguments for other cases.

Before we proceed to our results, we recall some relevant definitions. We consider finite, simple, and undirected graphs, and use standard terminology and notation. For a graph property Π , that is, Π is a set of graphs, the corresponding graph sandwich problem is the following.

Π -SANDWICH PROBLEM

Instance: A pair (G_1, G_2) of two graphs such that G_1 and G_2 have the same vertex set, and G_1 is a subgraph of G_2 .

Task: Determine a graph G with $G_1 \subseteq G \subseteq G_2$ and $G \in \Pi$, or conclude that no such graph exists.

Let \mathcal{F} be a set of graphs. A graph G is \mathcal{F} -free if no induced subgraph of G is in \mathcal{F} . Let $\overline{\mathcal{F}}$ be $\{\overline{F} : F \in \mathcal{F}\}$, where \overline{F} is the complement of a graph F . For two graphs G_1 and G_2 such that G_1 and G_2 have the same vertex set, and G_1 is a subgraph of G_2 , let $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ be the set of \mathcal{F} -free graphs G with $G_1 \subseteq G \subseteq G_2$.

Here is the type of problem we consider.

\mathcal{F} -FREE SANDWICH PROBLEM

Instance: A pair (G_1, G_2) of two graphs such that G_1 and G_2 have the same vertex set, and G_1 is a subgraph of G_2 .

Task: Determine a graph G in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$, or conclude that this set is empty.

The \mathcal{F} -FREE SANDWICH DECISION PROBLEM has the same input as the \mathcal{F} -FREE SANDWICH PROBLEM but the task is merely to decide whether $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty. It is easy to see that the \mathcal{F} -FREE SANDWICH PROBLEM can be solved in polynomial time if and only if the \mathcal{F} -FREE SANDWICH DECISION PROBLEM can. In fact, if $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty, then iteratively applying an efficient algorithm for the \mathcal{F} -FREE SANDWICH DECISION PROBLEM, one can determine in polynomial time an

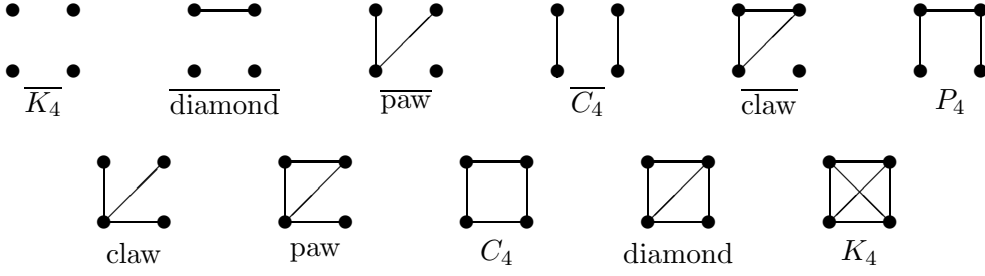


Figure 1: All graphs of order four.

edge-minimal graph G with $G_1 \subseteq G \subseteq G_2$ such that $\mathcal{SW}_{\mathcal{F}}(G_1, G)$ is still non-empty, and this graph G actually lies in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$. We collect some simple observations.

Observation 1.1 *Let \mathcal{F} be a set of graphs, and let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM.*

- (i) $\mathcal{SW}_{\mathcal{F}}(\overline{G_2}, \overline{G_1}) = \overline{\mathcal{SW}_{\mathcal{F}}(G_1, G_2)}$.
- (ii) *If all graphs in \mathcal{F} are connected, and $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty, then there is some graph G in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ such that the vertex sets of the components of G_1 are the same as the vertex sets of the components of G .*
- (iii) *If no graph in \mathcal{F} has a universal vertex, and u is a universal vertex in G_2 , then $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty if and only if $\mathcal{SW}_{\mathcal{F}}(G_1 - u, G_2 - u)$ is non-empty.*
- (iv) *If every graph F in \mathcal{F} has a unique \mathcal{F} -free supergraph F^* with $V(F) = V(F^*)$, then the \mathcal{F} -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: (i) This follows immediately from the definition.

(ii) Since the vertex set of each component of a graph G in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is the union of vertex sets of components of G_1 , and all edges of G between components of G_1 belong to G_2 , removing from G all such edges yields another graph in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ that has the desired property.

(iii) If $G \in \mathcal{SW}_{\mathcal{F}}(G_1, G_2)$, then $G - u \in \mathcal{SW}_{\mathcal{F}}(G_1 - u, G_2 - u)$, which implies the necessity. By the assumption on \mathcal{F} , adding a universal vertex to an \mathcal{F} -free graph yields an \mathcal{F} -free graph, which implies the sufficiency.

(iv) Starting with G_1 , and iteratively adding the uniquely determined sets of edges to every induced subgraph from \mathcal{F} using edges of G_2 yields a graph in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$. If, at some point, the graph G_2 does not contain the necessary edges, then $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is empty. \square

As said above, our goal is to study the complexity of the \mathcal{F} -FREE SANDWICH PROBLEM for sets \mathcal{F} containing two non-isomorphic graphs of order four. Figure 1 illustrates all such graphs together with the names we are using. By Observation 1.1(i), it suffices to consider the sets \mathcal{F} up to complementation. Note that P_4 is the only self-complementary graph of order four. Hence, up to complementation, there are 5 sets \mathcal{F} that contain P_4 . There are 10 sets \mathcal{F} containing two non-isomorphic graphs with less than four edges, and, up to complementation, there are 15 sets \mathcal{F} containing one graph with less than four edges and one graph with more than four edges. Altogether, the 30 choices for \mathcal{F} illustrated in Figure 2 represent all sets of two non-isomorphic graphs of order four up to complementation.

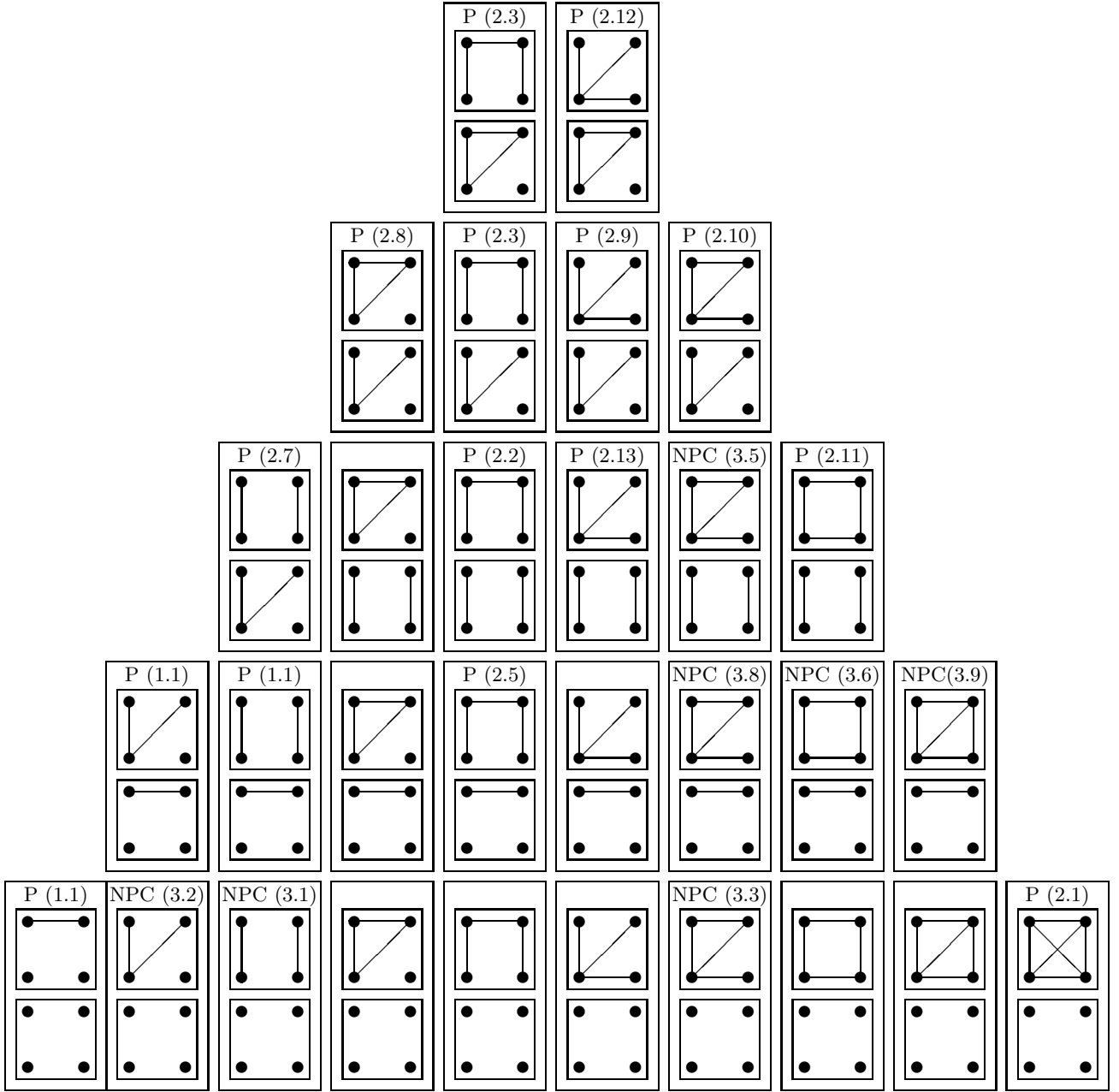


Figure 2: All 30 pairs of non-isomorphic graphs of order four up to complementation, together with the status of the corresponding sandwich decision problem, where “P” means “*polynomial time solvable*”, “NPC” means “*NP-complete*”, and the number in the bracket is the reference number of the corresponding result within this paper.

In Sections 2 and 3, we collect our positive and negative results, respectively. In a final section, we conclude with some comments on the open cases.

2 Some Tractable Cases

We present our positive results in an order of roughly increasing difficulty.

Observation 1.1(i) and (iv) imply that the \mathcal{F} -FREE SANDWICH PROBLEM can be solved in polynomial time if \mathcal{F} or $\overline{\mathcal{F}}$ is one of the sets

$$\{P_3\}, \{\text{diamond}, K_4\}, \{\text{diamond}, C_4\}, \text{ or } \{\text{diamond}, \text{paw}\}.$$

In order to understand the complexity of the \mathcal{F} -FREE SANDWICH PROBLEM, if \mathcal{F} is as in Observation 1.1(ii), then it suffices to consider instances (G_1, G_2) such that G_1 is connected, and, if \mathcal{F} is as in Observation 1.1(iii), then it suffices to consider instances (G_1, G_2) such that G_2 has no universal vertex; otherwise, in both cases some simple algorithmic reduction applies.

For positive integers s and t , let $R(s, t)$ be the *Ramsey number*, in particular, every graph of order $R(s, t)$ contains an induced K_s or \overline{K}_t .

Theorem 2.1 *The $\{K_4, \overline{K}_4\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: No instance (G_1, G_2) of the $\{K_4, \overline{K}_4\}$ -FREE SANDWICH PROBLEM for which G_1 has order at least $R(4, 4)$ has a solution. Instances (G_1, G_2) for which G_1 has order less than $R(4, 4)$ can be solved in constant time. \square

It is well-known [3] that, for every P_4 -free graph G of order at least 2, either G or \overline{G} is disconnected.

Theorem 2.2 *The $\{P_4, C_4\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: Let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM for $\mathcal{F} = \{P_4, C_4\}$. By Observation 1.1(ii) and (iii), we may assume that G_1 is connected, and that G_2 has no universal vertex. Suppose that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some graph G . Let u is a vertex of maximum degree in G . Since u is not universal, there is an induced path uvw of order 3 in G . Since w is a neighbor of v but not of u , and u has at least as many neighbors as v , there is a vertex x that is a neighbor of u but not of v . Nevertheless, the subgraph $G[\{u, v, w, x\}]$ of G induced by $\{u, v, w, x\}$ is either P_4 or C_4 , which is a contradiction. Hence, either one of the two algorithmic reductions corresponding to Observation 1.1(ii) and (iii) applies, or $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is necessarily empty. \square

The following result concerns the two cases $\mathcal{F} = \{P_4, \overline{\text{paw}}\}$ and $\mathcal{F} = \{P_4, \overline{\text{claw}}\}$.

Theorem 2.3 *The $\{P_4, K_1 \cup F_2\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time for $F_2 \in \{K_3, P_3\}$.*

Proof: Let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM for $\mathcal{F} = \{P_4, K_1 \cup F_2\}$, where $n(G_1) \geq 2$. If $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some disconnected graph G , then G is $\{P_4, F_2\}$ -free. Since, by Observation 1.1(iv), the $\{P_4, F_2\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time, this possibility can be checked in polynomial time. Hence, we may assume that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains no disconnected graph. If $\overline{G_2}$ is connected, then, since, for every graph G in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$, the graph \overline{G} is P_4 -free and contains $\overline{G_2}$, $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is empty. Hence, we may assume that $\overline{G_2}$ is disconnected. Note that, if H is the join of two graphs H_1 and H_2 , then every induced P_4 or $K_1 \cup F_2$ in H is completely contained either in H_1 or in H_2 . Hence, if K is the vertex set of some component of $\overline{G_2}$, then $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty if and only if $\mathcal{SW}_{\mathcal{F}}(G_1[K], G_2[K])$ and $\mathcal{SW}_{\mathcal{F}}(G_1 - K, G_2 - K)$ are both non-empty, that is, in polynomial time, one can reduce the instance (G_1, G_2) to two smaller instances (G'_1, G'_2) and (G''_1, G''_2) such that $n(G_1) = n(G'_1) + n(G''_1)$, which implies the desired statement. \square

Deciding the existence of a complete bipartite sandwich can easily be reduced to 2SAT [12, 18]. We give a different argument leading to a simpler algorithm.

Lemma 2.4 *If Π is the set of all complete bipartite graphs, then the Π -SANDWICH PROBLEM can be solved in polynomial time.*

Proof: Let (G_1, G_2) be an instance of the Π -SANDWICH PROBLEM. Clearly, we may assume that all components K_1, \dots, K_p of G_1 are bipartite. Let K_i have the partite sets A_i and B_i for $i \in [p]$. Initialize a set \mathcal{P} as $\{\{A_1, B_1\}, \dots, \{A_p, B_p\}\}$, and, iteratively and as long as possible, whenever \mathcal{P} contains two distinct sets $\{X, Y\}$ and $\{X', Y'\}$ such that, in G_2 , some vertex in X is non-adjacent to some vertex in X' , then replace $\{X, Y\}$ and $\{X', Y'\}$ within \mathcal{P} by $\{X \cup X', Y \cup Y'\}$; breaking ties arbitrarily. Note that $X \cup X'$ is the union of partite sets of components of G_1 that necessarily belong to the same partite set of any solution. When \mathcal{P} no longer changes, then, for every two distinct sets $\{X, Y\}$ and $\{X', Y'\}$ in \mathcal{P} , the graph G_2 contains all edges between $X \cup Y$ and $X' \cup Y'$. Therefore, if the final \mathcal{P} contains the sets $\{X_1, Y_1\}, \dots, \{X_q, Y_q\}$, then there is a complete bipartite graph G with $G_1 \subseteq G \subseteq G_2$ if and only if G_2 contains all edges between X_i and Y_i for every $i \in [q]$. Furthermore, such a graph G can easily be determined. \square

Theorem 2.5 *The $\{P_4, \overline{\text{diamond}}\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: Let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM for $\mathcal{F} = \{P_4, \overline{\text{diamond}}\}$, where $m(G_1) > 0$.

Suppose that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some disconnected graph G . Since G is P_4 -free, \overline{G} is a connected graph in $\mathcal{SW}_{\overline{\mathcal{F}}}(\overline{G_2}, \overline{G_1})$. Let \overline{G} be the join of the two non-empty graphs $\overline{G_L}$ and $\overline{G_R}$. Since \overline{G} is diamond-free, the two graphs $\overline{G_L}$ and $\overline{G_R}$ are P_3 -free, that is, they are the unions of k_L and k_R complete graphs, respectively. Since G has at least one edge, we may assume, by symmetry, that $k_R \geq 2$. Since \overline{G} is diamond-free, this implies that all vertices of $\overline{G_L}$ are isolated. If $k_L = 1$, then $\overline{G_L}$ consists of a universal vertex u_L of \overline{G} . Since the $\{P_3\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time, considering all $n(G_1)$ choices for u_L , one can check in polynomial time whether $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains such a graph G . Hence, we may assume that $k_L \geq 2$. Since \overline{G} is diamond-free, this implies that all vertices of $\overline{G_R}$ are isolated, that is, \overline{G} is a complete bipartite graph. By Lemma 2.4, one can check in polynomial time whether $\mathcal{SW}_{\overline{\mathcal{F}}}(\overline{G_2}, \overline{G_1})$ contains a complete bipartite graph. Altogether, it follows that one can check in polynomial time whether $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some disconnected graph. Hence, we may assume that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains no disconnected graph.

If $\overline{G_2}$ is connected, then, similarly as in the proof of Theorem 2.3, $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is empty. Hence, we may assume that $\overline{G_2}$ is disconnected. If K is the vertex set of some component of $\overline{G_2}$, then, similarly as in the proof of Theorem 2.3, $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty if and only if $\mathcal{SW}_{\mathcal{F}}(G_1[K], G_2[K])$ and $\mathcal{SW}_{\mathcal{F}}(G_1 - K, G_2 - K)$ are both non-empty, which implies the desired statement. \square

Our next few results involve the paw, and the following result of Olariu is quite useful.

Lemma 2.6 (Olariu [17]) *A connected graph is paw-free if and only if it is triangle-free or $\overline{P_3}$ -free.*

Theorem 2.7 *The $\{\text{paw}, C_4\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: Let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM for $\mathcal{F} = \{\text{paw}, C_4\}$. By Observation 1.1(ii), we may assume that G_1 is connected.

Suppose that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some graph G . By Lemma 2.6, G is triangle-free or $\overline{P_3}$ -free. Since G is $\{K_3, C_4\}$ -free if and only if $G = G_1$, and G_1 is $\{K_3, C_4\}$ -free, we may assume that G is $\{\overline{P_3}, C_4\}$ -free. This implies that G is a complete multipartite graph with at most one partite set of order more than 1, that is, \overline{G} has at most one edge. This implies that $\overline{G_2}$ has at most one edge, and, hence, that G_2 is $\{\overline{P_3}, C_4\}$ -free. Altogether, if $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty, then G_1 or G_2 belongs to this set. \square

Theorem 2.8 *The $\{\text{paw}, \text{claw}\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: Let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM for $\mathcal{F} = \{\text{paw}, \text{claw}\}$. By Observation 1.1(ii), we may assume that G_1 is connected.

Suppose that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some graph G . By Lemma 2.6, G is triangle-free or $\overline{P_3}$ -free. Since G is $\{K_3, \text{claw}\}$ -free if and only if $G = G_1$, and G_1 is $\{K_3, \text{claw}\}$ -free, we may assume that G is $\{\overline{P_3}, \text{claw}\}$ -free. This implies that G is a complete multipartite graph such that each partite set contains at most two vertices, that is, \overline{G} has maximum degree at most 1. This implies that $\overline{G_2}$ has maximum degree at most 1, and, hence, that G_2 is $\{\overline{P_3}, \text{claw}\}$ -free. Similarly as in the proof of Theorem 2.7, if $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty, then G_1 or G_2 belong to this set. \square

Theorem 2.9 *The $\{\text{paw}, \overline{\text{claw}}\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: Let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM for $\mathcal{F} = \{\text{paw}, \overline{\text{claw}}\}$.

Suppose that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some graph G . If G is triangle-free, then G_1 is also triangle-free, and, hence, lies in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$. Hence, we may assume that G is not triangle-free. Since G is $\overline{\text{claw}}$ -free, G is connected. By Lemma 2.6, G is $\overline{P_3}$ -free. Since a graph is $\{\overline{P_3}, \overline{\text{claw}}\}$ -free if and only if it is $\overline{P_3}$ -free, and the $\{\overline{P_3}\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time, one can check in polynomial time whether $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains such a graph. \square

Theorem 2.10 *The $\{\text{paw}, \overline{\text{paw}}\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: Let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM for $\mathcal{F} = \{\text{paw}, \overline{\text{paw}}\}$.

Suppose that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some graph G . Since $\overline{\mathcal{F}} = \mathcal{F}$, we may assume, by Observation 1.1, that G is connected. By Lemma 2.6, this implies that G is triangle-free or $\overline{P_3}$ -free. Since a graph is $\{\overline{\text{paw}}, \overline{P_3}\}$ -free if and only if it is $\overline{P_3}$ -free, and the $\{\overline{P_3}\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time, one can check in polynomial time whether $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains a $\{\overline{\text{paw}}, \overline{P_3}\}$ -free graph. Hence, we may assume that G is $\{\overline{\text{paw}}, K_3\}$ -free.

By Lemma 2.4, we may assume that G is not complete bipartite. If the maximum degree of G is at most 2, then G has at most 5 vertices. Hence, we may assume that G has maximum degree at least 3. Let u be a vertex of maximum degree. Let B be the neighborhood of u . Let A be the set of vertices whose neighborhood is B . Since G is triangle-free, $G[A \cup B]$ is a complete bipartite graph with partite sets A and B . Since G is connected but not complete bipartite, some vertex in B has a neighbor w outside of A . By the definition of A , w has a non-neighbor v in B . Since G is $\overline{\text{paw}}$ -free, v is the only non-neighbor of w in B . Now, v , w , and two further vertices from B induce a $\overline{\text{paw}}$, which is a contradiction. \square

Our next result relies on Maffray and Preissmann's [15] characterization of pseudo-split graphs, and Golumbic, Kaplan, and Shamir's [12] algorithm for the split sandwich problem.

Theorem 2.11 *The $\{C_4, \overline{C_4}\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: Let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM for $\mathcal{F} = \{C_4, \overline{C_4}\}$. Suppose that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some graph G . By a result of Maffray and Preissmann [15], there is a set C of at most five vertices such that $G - C$ is a split graph. Considering the $O(n(G_1)^5)$ choices for C , and applying the polynomial time algorithm of Golumbic, Kaplan, and Shamir [12] to $G - C$, one can decide in polynomial time whether $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty. \square

The next proof uses a result of Brandstädt and Mahfud [4] concerning prime $\{\text{claw}, \overline{\text{claw}}\}$ -free graphs.

Theorem 2.12 *The $\{\text{claw}, \overline{\text{claw}}\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: Let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM for $\mathcal{F} = \{\text{claw}, \overline{\text{claw}}\}$, with $n(G_1) \geq 10$. Suppose that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some graph G . We will show that either G or \overline{G} has maximum degree at most 2, that is, it is the union of paths and cycles. If G has maximum degree at most 2, then G_1 belong to $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$, and, if \overline{G} has maximum degree at most 2, then G_2 belong to $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$, which clearly implies the desired statement.

If G is disconnected, then G is $\{\text{claw}, K_3\}$ -free, which clearly implies that G has maximum degree at most 2. Hence, by symmetry, we may assume that G and \overline{G} are both connected. If G is prime, then Brandstädt and Mahfud [4] showed that G or \overline{G} has maximum degree at most 2. Hence, we may assume that G contains a homogeneous set U of vertices, that is, $2 \leq |U| \leq n(G) - 1$, and $V(G) = U \cup A \cup B$, where A is the set of vertices in $V(G) \setminus U$ that are adjacent to every vertex in U , and, B is the set of vertices in $V(G) \setminus U$ that are adjacent to no vertex in U . Since G and \overline{G} are both connected, there are vertices a and a' in A , and, b and b' in B such that a and b are adjacent, and, a' and b' are non-adjacent. Let u and u' be two vertices in U . If u and u' are adjacent, then $G[\{u, u', a', b'\}]$ is a $\overline{\text{claw}}$, and, if u and u' are not adjacent, then $G[\{u, u', a, b\}]$ is a claw, which completes the proof. \square

Theorem 2.13 *The $\{\text{claw}, \overline{C_4}\}$ -FREE SANDWICH PROBLEM can be solved in polynomial time.*

Proof: Let (G_1, G_2) be an instance of the \mathcal{F} -FREE SANDWICH PROBLEM for $\mathcal{F} = \{\overline{\text{claw}}, C_4\}$, which, by Observation 1.1(i), is equivalent to the $\{\text{claw}, \overline{C_4}\}$ -FREE SANDWICH PROBLEM. By Observation 1.1(iii), we may assume that G_2 has no universal vertex. If G_1 is $\{K_3, C_4\}$ -free, then $G_1 \in \mathcal{SW}_{\mathcal{F}}(G_1, G_2)$. Hence, we may assume that G_1 contains an induced K_3 or C_4 , which implies that every graph in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains a triangle, and, hence, in view of $\overline{\text{claw}}$, is connected.

Suppose that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some graph G such that not all vertices of G lie on triangles. Let T be the set of vertices of G that lie on triangles, and let $R = V(G) \setminus T$, in particular, T and R are both non-empty. Since G is $\overline{\text{claw}}$ -free, every vertex in R has a neighbor in T . If some vertex u in R has two neighbors v and w in T , then, since u does not lie on a triangle, v and w are not adjacent. Let vxy be a triangle that contains v . Since G is $\overline{\text{claw}}$ -free, w is adjacent to x or y , and $uvwxu$ or $uvywu$ is a C_4 , which is a contradiction. Hence, every vertex in R has exactly one neighbor in T . Let v_1, \dots, v_p be the vertices in T that are the neighbor of some vertex in R . Since R is not empty, we have $p \geq 1$. Let u_i be a neighbor of v_i in R for $i \in [p]$. Since G is $\overline{\text{claw}}$ -free, every triangle of G contains all vertices v_1, \dots, v_p , which implies $p \leq 3$. If xy is an edge between two vertices in R , then x and y have different neighbors, say v_i and v_j , among v_1, \dots, v_p . Since v_i and v_j both belong to every triangle, they are adjacent, and the vertices x, y, v_i , and v_j form a C_4 , which is a contradiction. Hence, R is independent. If $p = 1$, then v_1 is a universal vertex of G , and, hence, also of G_2 , which is a contradiction. Hence, $p \in \{2, 3\}$. If $p = 3$, then $T = \{v_1, v_2, v_3\}$, that is, G contains exactly one triangle, and considering the $O(n(G_1)^3)$ choices for v_1, v_2 , and v_3 , it is possible to check in polynomial time whether $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains such a graph. If $p = 2$, then $V(G) \setminus \{v_1, v_2\}$ is independent, which implies that considering the $O(n(G_1)^2)$ choices for v_1 and v_2 , it is possible to check in polynomial time whether $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains such a graph. Altogether, it follows that one can check in polynomial time whether $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains some graph G such that not all vertices of G lie on triangles. Hence, we may assume that all vertices of every graph in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ lie on triangles.

Suppose that G is an edge-maximal graph in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$. If G contains $\overline{K_1 \cup C_4}$ as an induced subgraph, then the vertex of degree 4 in $\overline{K_1 \cup C_4}$ is universal in G , which is a contradiction. Hence,

G is $\overline{K_1 \cup C_4}$ -free. Our next goal is to show that G contains the diamond as an induced subgraph. Suppose, for a contradiction, that G is diamond-free. Since G has no universal vertex, there is a triangle uvw in G as well as a vertex x distinct from u, v , and w such that x is not adjacent to u . Since G is diamond-free, we may assume that x is adjacent to v but not to w . Since x lies on some triangle, it has a neighbor y outside of $\{u, v, w\}$. Since G is $\{\overline{\text{claw}}, C_4\}$ -free, y is adjacent to v . Since G is $\overline{K_1 \cup C_4}$ -free, y is adjacent to u or w , which yields a diamond in both cases. Hence, G contains a diamond.

Let u, v, w , and x induce a diamond in G such that u and x are not adjacent. Clearly, every vertex of G is adjacent to v or w or both. Since G_2 has no universal vertex, the two sets $N_v = N_G(v) \setminus N_G[w]$ and $N_w = N_G(w) \setminus N_G[v]$ are both not empty. Since G is $\{\overline{\text{claw}}, C_4\}$ -free, $N_v \cup N_w$ is independent. Let R be the set of vertices in $N_G(v) \cap N_G(w)$ that have a neighbor in $N_v \cup N_w$, and let $S = (N_G(v) \cap N_G(w)) \setminus R$. By definition, G contains no edge between $N_v \cup N_w$ and S . Since every vertex of G lies on a triangle, every vertex in $N_v \cup N_w$ has a neighbor in R , in particular, R is not empty. Furthermore, since G is $\overline{\text{claw}}$ -free, $N_v \cup N_w$ is completely joined to R . If R contains two non-adjacent vertices x and y , then x, y , a vertex from N_v , and a vertex from N_w form a C_4 . Hence, R is a clique. Since G is $\overline{\text{claw}}$ -free, S is independent. Since G is $\overline{\text{claw}}$ -free, every vertex in S has at most one non-neighbor in R . Since adding an edge between N_v and w or between N_w and v does not create an induced subgraph $\overline{\text{claw}}$ or C_4 , the edge-maximality of G implies that G contains all edges of G_2 between $\{v, w\}$ and $V(G_1) \setminus \{v, w\}$. Similarly, since adding an edge between R and S does not create an induced subgraph $\overline{\text{claw}}$ or C_4 , the edge-maximality of G implies that G contains all edges of G_2 between R and S . Altogether, it follows that there is an edge vw of G_2 , and a partition of the set $N_{G_2}(v) \cap N_{G_2}(w)$ into two sets R and S such that

- (i) the two sets $N_v = N_{G_2}(v) \setminus N_{G_2}[w]$ and $N_w = N_{G_2}(w) \setminus N_{G_2}[v]$ are non-empty,

$$V(G_1) = \{v, w\} \cup N_v \cup N_w \cup (N_{G_2}(v) \cap N_{G_2}(w)),$$

$N_v \cup N_w$ is independent in G_1 , and,

- (ii) R is a clique in G_2 ,

S is independent in G_1 ,

G_2 contains all possible edges between R and $N_v \cup N_w$,

G_1 contains no edge between S and $N_v \cup N_w$, and,

in G_2 , every vertex in S has at most one non-neighbor in R .

Conversely, if there is an edge vw of G_2 , and a partition of the set $N_{G_2}(v) \cap N_{G_2}(w)$ into two sets R and S such that (i) and (ii) are satisfied, then it is easy to see that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty.

Let vw be an edge of G_2 , and let N_v and N_w be as in (i). Clearly, deciding whether (i) is satisfied can be done in polynomial time. Furthermore, we now explain how to decide in polynomial time using 2SAT whether $N_{G_2}(v) \cap N_{G_2}(w)$ has a partition into two sets R and S that satisfies (ii). Let $X = N_{G_2}(v) \cap N_{G_2}(w)$. For every vertex x in X , we introduce a boolean variable x , which should be true if x is in R , and false if x is in S . Now, we construct a 2SAT formula f as follows.

- For every two vertices x and y in X that are non-adjacent in G_2 , we add to f the clause $\bar{x} \vee \bar{y}$, reflecting that R is a clique in G_2 .
- For every two vertices x and y in X that are adjacent in G_1 , we add to f the clause $x \vee y$, reflecting that S is independent in G_1 .

- For every vertex x in X that is non-adjacent in G_2 to some vertex in $N_v \cup N_w$, we add to f the clause \bar{x} , reflecting that G_2 contains all possible edges between R and $N_v \cup N_w$.
- For every vertex x in X that is adjacent in G_1 to some vertex in $N_v \cup N_w$, we add to f the clause x , reflecting that G_1 contains no edge between S and $N_v \cup N_w$.
- For every two vertices x and y in X , for which there is a third vertex z in X such that x and y are both non-adjacent in G_2 to z , we add to f the clause $\bar{x} \vee \bar{y}$, reflecting that, in G_2 , every vertex in S has at most one non-neighbor in R .

It is easy to see that f is satisfiable if and only if $N_{G_2}(v) \cap N_{G_2}(w)$ has the desired partition. Therefore, considering all $O(n(G_1)^2)$ edges vw of G_2 , one can determine in polynomial time whether $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty. \square

3 Some Hard Cases

For every finite set \mathcal{F} of graphs, the \mathcal{F} -FREE SANDWICH DECISION PROBLEM clearly belongs to NP.

Dantas, de Figueiredo, da Silva, and Teixeira [7] showed that the $\{C_4\}$ -FREE SANDWICH DECISION PROBLEM is NP-complete. Considering the proof of the corresponding result (Theorem 1 in [7]), it is easy to see that the very same proof yields the following result.

Theorem 3.1 *The $\{C_4, K_4\}$ -FREE SANDWICH DECISION PROBLEM is NP-complete.*

Our next two results rely on the hardness of deciding 3-colorability.

Theorem 3.2 *The $\{\text{paw}, K_4\}$ -FREE SANDWICH DECISION PROBLEM is NP-complete.*

Proof: By Lemma 2.6, a connected graph G that contains a triangle is $\{\text{paw}, K_4\}$ -free if and only if it is $\{\overline{P_3}, K_4\}$ -free if and only if it is complete multipartite with at most three partite sets. Furthermore, a graph is 3-colorable if and only if it has a complete multipartite supergraph with at most three partite sets. Therefore, a given connected graph G that contains a triangle is 3-colorable if and only if $\mathcal{SW}_{\mathcal{F}}(G, K_{n(G)})$ is non-empty for $\mathcal{F} = \{\text{paw}, K_4\}$. Since deciding 3-colorability for such graphs is NP-complete, the desired statement follows. \square

Theorem 3.3 *The $\{\text{paw}, \overline{K_4}\}$ -FREE SANDWICH DECISION PROBLEM is NP-complete.*

Proof: Let H be a graph. Let G_1 arise from H by adding three disjoint sets X , Y , and Z each containing $R(3, 4)$ new vertices, and adding all edges between X and Y , between X and Z , and between Y and Z . Let G_2 arise from G_1 by adding all edges between $V(H)$ and $X \cup Y \cup Z$, and by adding all edges of \overline{H} . Note that H is 3-colorable if and only if G_1 is 3-colorable. Furthermore, for every 3-coloring of G_1 , the three sets X , Y , and Z are subsets of different color classes. Hence, if G_1 is 3-colorable, then G_2 contains a complete multipartite supergraph of G_1 with three partite sets. We will show that H is 3-colorable if and only if $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is non-empty for $\mathcal{F} = \{\overline{\text{paw}}, K_4\}$. By Observation 1.1(i), this implies the desired statement.

First, suppose that H is 3-colorable. As observed above, G_2 contains a complete multipartite supergraph G of G_1 with three partite sets. Since G is in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$, the necessity follows. For the proof of the sufficiency, suppose that G is in $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$. First, suppose that \overline{G} is connected. By Lemma 2.6, \overline{G} is $\{K_3, \overline{K_4}\}$ -free or $\{\overline{P_3}, \overline{K_4}\}$ -free. In the first case, $n(G_1) \leq R(3, 4)$, which is a

contradiction. In the second case, \overline{G} is a complete multipartite graph with partite sets of order at most 3, which implies the contradiction $\Delta(G_1) \leq \Delta(G) \leq 2$. Hence, \overline{G} is disconnected. Since \overline{G} is $\overline{K_4}$ -free, \overline{G} has either two or three components. First, suppose that \overline{G} has two components. Since \overline{G} is $\{\text{paw}, \overline{K_4}\}$ -free, one component, say \overline{K} , of \overline{G} is a clique, and, by Lemma 2.6, the other component, say $\overline{K'}$, of \overline{G} is $\{K_3, \overline{K_3}\}$ -free or $\{\overline{P_3}, \overline{K_3}\}$ -free. If $\overline{K'}$ is $\{K_3, \overline{K_3}\}$ -free, then $n(\overline{K'}) \leq R(3, 3)$. Since $V(\overline{K})$ is independent in G , we may assume, by symmetry between X , Y , and Z , that $V(\overline{K'})$ contains a vertex of X . Since G contains all edges between $V(\overline{K})$ and $V(\overline{K'})$, this implies that the independent set X is contained in $V(\overline{K})$, which is impossible because $|X| = R(3, 4) > n(\overline{K'})$. Hence, $\overline{K'}$ is $\{\overline{P_3}, \overline{K_3}\}$ -free. This implies that $\overline{K'}$ is a complete multipartite graph with partite sets of order at most 2 and, hence, G , G_1 , and H are 3-colorable. Finally, suppose that \overline{G} has three components. It follows that each component of \overline{G} is complete, that is, G is a complete multipartite graph with three partite sets. Therefore, G , and, thus, also G_1 and H are 3-colorable. \square

Our next two results rely on related results concerning $\{\overline{C_4}, C_3, C_5\}$ -free graphs, which are known as *chain graphs* or *difference graphs* (cf. Theorem 2.4.4 in [16]). Clearly, chain graphs are bipartite. While the next two proofs are based on essentially the same approach, we argue from first principle for the first, and rely on results about prime $\{\text{diamond}, \overline{C_4}\}$ -free graphs [1] for the second.

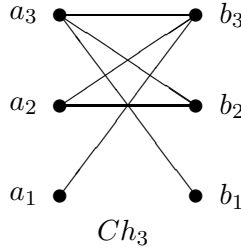


Figure 3: The chain graph Ch_3 .

Lemma 3.4 *Let $\mathcal{F} = \{\text{paw}, \overline{C_4}\}$. Let G_1 be the disjoint union of a graph G and the graph Ch_3 in Figure 3, and let G_2 arise from G_1 by adding all edges between $V(G)$ and $V(Ch_3)$.*

(i) *If G is C_3 or C_5 , then $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ is empty.*

(ii) *If G is a chain graph, then $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains a chain graph.*

Proof: (i) We only give details for the case that G is a triangle xyz . The case that G is a C_5 can be settled similarly.

For a contradiction, suppose that $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$ contains a graph H . Since H is $\overline{C_4}$ -free, considering the edges a_3b_3 and xy , we may assume, by symmetry, that a_3 and x are adjacent. Since H is paw-free, considering a_3 and the triangle xyz , we may assume, by symmetry, that a_3 is adjacent to y . Since H is paw-free, considering b_3 and the triangle a_3xy , we may assume, by symmetry, that b_3 is adjacent to y . Since H is paw-free, considering any of the vertices a_1 , a_2 , and b_2 together with the triangle a_3b_3y , we obtain that a_1 , a_2 , and b_2 are adjacent to y . Now, $H[\{a_1, a_2, b_2, y\}]$ is a paw, which is a contradiction.

(ii) Let G have the partite sets A and B . Let G' arise from the disjoint union of G and Ch_3 by adding all edges between $V(G)$ and $\{a_1, a_2, a_3\}$. Clearly, the sets $A' = A \cup \{a_1, a_2, a_3\}$ and $B' = B \cup \{b_1, b_2, b_3\}$ form a bipartition of G' . Suppose that G' contains an induced $\overline{C_4}$ with the two edges ab and $a'b'$,

where $a, a' \in A'$. If $a \in \{a_1, a_2, a_3\}$, then, in view of the edges between $V(G)$ and $\{a_1, a_2, a_3\}$, it follows that $b' \in \{b_1, b_2, b_3\}$, which implies that $a' \in \{a_1, a_2, a_3\}$, and, hence, by symmetry, $b \in \{b_1, b_2, b_3\}$. Nevertheless, since Ch_3 is a chain graph, this is a contradiction. If $a \in A$, then, in view of the structure of G' , it follows that $b \in B$, which implies that $a' \in A$, and, hence, by symmetry, b' . Nevertheless, since G is a chain graph, this is a contradiction. Altogether, G' is a bipartite $\overline{C_4}$ -free graph, that is, G' is a chain graph. By construction, G' belongs to $\mathcal{SW}_{\mathcal{F}}(G_1, G_2)$. \square

Theorem 3.5 *The $\{\text{paw}, \overline{C_4}\}$ -FREE SANDWICH DECISION PROBLEM is NP-complete.*

Proof: Let $\mathcal{F} = \{\text{paw}, \overline{C_4}\}$, and let Π be the set of all chain graphs. In [9] Dantas, Figueiredo, Golumbic, Klein, and Maffray describe a polynomial reduction of an instance f of an NP-complete variant of SATISFIABILITY to an instance (G_1, G_2) of the Π -SANDWICH DECISION PROBLEM; the decision variant of the Π -SANDWICH PROBLEM. Let G'_1 be the disjoint union of G_1 and the graph Ch_3 , and, let G'_2 arise from the disjoint union of G_2 and the graph Ch_3 by adding all edges between $V(G_2)$ and $V(Ch_3)$.

If there is a chain graph G with $G_1 \subseteq G \subseteq G_2$, then, by Lemma 3.4(ii), $\mathcal{SW}_{\mathcal{F}}(G'_1, G'_2)$ is non-empty. Conversely, if $\mathcal{SW}_{\mathcal{F}}(G'_1, G'_2)$ contains some graph G' , then, by Lemma 3.4(i), the graph $G = G' - V(Ch_3)$ is $\{\overline{C_4}, C_3, C_5\}$ -free, that is, G is a chain graph. By construction, $G_1 \subseteq G \subseteq G_2$. Altogether, we obtain a polynomial reduction of some NP-complete problem to the $\{\text{paw}, \overline{C_4}\}$ -FREE SANDWICH DECISION PROBLEM, which completes the proof. \square

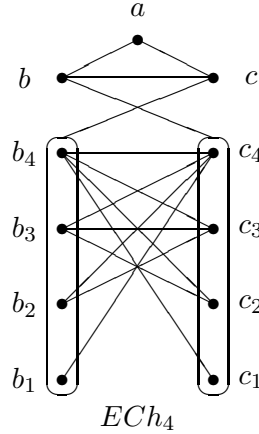


Figure 4: The graph ECh_4 ; the vertex b is adjacent to all vertices in the independent set $\{c_1, c_2, c_3, c_4\}$, and, the vertex c is adjacent to all vertices in the independent set $\{b_1, b_2, b_3, b_4\}$.

Theorem 3.6 *The $\{\text{diamond}, \overline{C_4}\}$ -FREE SANDWICH DECISION PROBLEM is NP-complete.*

Proof: Let $\mathcal{F} = \{\text{diamond}, \overline{C_4}\}$, and let Π be the set of all chain graphs. In [9] Dantas, Figueiredo, Golumbic, Klein, and Maffray describe a polynomial reduction of an instance f of an NP-complete variant of SATISFIABILITY to an instance (G_1, G_2) of the Π -SANDWICH DECISION PROBLEM, where the edges of G_1 form a perfect matching. We proceed similarly as in the proof of Theorem 3.5, that is, we describe a polynomial reduction of (G_1, G_2) to an instance (G'_1, G'_2) of the \mathcal{F} -FREE SANDWICH DECISION PROBLEM.

Let G'_1 be the disjoint union of G_1 and the graph ECh_4 in Figure 4, and, let G'_2 arise from the disjoint union of G_2 and the graph ECh_4 by adding all edges between $V(G_2)$ and $V(ECh_4) \setminus \{a\}$.

First, suppose that there is a chain graph G with $G_1 \subseteq G \subseteq G_2$. Since G_1 has no isolated vertices, and G is $\overline{C_4}$ -free, it follows that G is connected. Let B and C be the partite sets of G . Let G' arise

from the disjoint union of G_1 and the graph ECh_4 by adding all edges between $\{b, b_1, b_2, b_3, b_4\}$ and C as well as all edges between $\{c\}$ and B . Similarly as in the proof of Lemma 3.4(ii), it follows that $G' \in \mathcal{SW}_{\mathcal{F}}(G'_1, G'_2)$.

Next, suppose that $\mathcal{SW}_{\mathcal{F}}(G'_1, G'_2)$ contains some graph G' . Since G'_1 has no isolated vertices, and G' is $\overline{C_4}$ -free, it follows that G' is connected. In view of the edges of ECh_4 , and, since the vertex a has no neighbor in $V(G_1)$, also $\overline{G'}$ is connected. Suppose that U is a homogeneous set of G' . Let A be the set of vertices in $V(G) \setminus U$ that are adjacent to every vertex in U , and, let N be the set of vertices in $V(G) \setminus U$ that are adjacent to no vertex in U . Since G' and $\overline{G'}$ are connected, both sets A and N are non-empty. Since G' is connected and diamond-free, the graph $G'[U]$ is P_3 -free. If U is neither independent nor a clique, then, since G' is diamond-free, it follows that A contains only one vertex, and, since $\overline{G'}$ is $\overline{C_4}$ -free, it follows that N is independent. In this case, since $\overline{G'}$ is connected, the unique vertex in A is universal in G' , which is a contradiction, because G_2 has no universal vertex. If U is a clique, then, since G' is \mathcal{F} -free, it follows that A is a clique and N is independent. Since G' is diamond-free, every vertex in N has exactly one neighbor in A . In particular, it follows that the vertices of degree at least 2 in G' form a clique, which is a contradiction in view of the two non-adjacent vertices b_2 and c_2 , which have degree at least 2 in G_1 . Hence, every homogeneous set of G' is independent. This easily implies that $\{a\}$, $\{b\}$, and $\{c\}$ are maximal homogeneous sets of G' , and, that all remaining vertices of ECh_4 belong to distinct maximal homogeneous sets of G' . This implies that ECh_4 as an induced subgraph of the characteristic graph G^* of G' ; in particular, the order of G^* is at least 11. By a result of Brandstädt [1], G^* is

- (i) either a thin spider, that is, $V(G^*)$ can be partitioned into a clique C and a stable set S , and the edges between C and S form a matching that covers all of S and all but at most one vertex of C ,
- (ii) or G^* arises from the disjoint union of a triangle $a^*b^*c^*$ and a connected chain graph with partite sets B^* and C^* by adding all edges between b^* and C^* as well as all edges between c^* and B^* .

If G^* is a thin spider, then, since the vertices that belong to maximal homogeneous sets represented by S have independent neighborhoods, the vertices a , b , and c of ECh_4 correspond to maximal homogeneous sets in C . Since b_4 is non-adjacent to c , and, c_4 is non-adjacent to b , the vertices b_4 and c_4 lie in S , which is a contradiction, since S is independent. Hence, G^* is as in (ii). Since a is the only vertex of G' whose removal yields a bipartite graph without creating a new vertex of degree 1, it follows that $\{a\} = a^*$, and, by symmetry, $\{b\} = b^*$ and $\{c\} = c^*$. Since $G^* - \{a^*, b^*, c^*\}$ is a chain graph, and, every homogeneous set of G' is independent, it follows that $G' - \{a, b, c\}$, and, hence also $G = G' - V(ECh_4)$ is a chain graph. Since, by construction, $G_1 \subseteq G \subseteq G_2$, this completes the proof. \square

For our last two hardness results, we prove the following auxiliary hardness result, which might be of independent interest.

Theorem 3.7 *Let Π be the set of all bipartite graphs G with a bipartition A and B such that every vertex in A has at most one non-neighbor in B , and, every vertex in B has at most one non-neighbor in A .*

The Π -SANDWICH DECISION PROBLEM is NP-complete.

Proof: The considered decision problem is clearly in NP. In order to complete the proof, we describe a polynomial reduction of the well known NP-complete ONE-IN-THREE 3SAT (cf. [LO4] in [11]) to the

II-SANDWICH DECISION PROBLEM. Therefore, let f be an instance of ONE-IN-THREE 3SAT consisting of the clauses C_1, \dots, C_m over the boolean variables x_1, \dots, x_n . We construct an instance (G_1, G_2) of the II-SANDWICH DECISION PROBLEM whose size is polynomially bounded in terms of n and m such that f is a ‘yes’-instance of ONE-IN-THREE 3SAT if and only if (G_1, G_2) is a ‘yes’-instance of the II-SANDWICH DECISION PROBLEM.

Starting with the empty graph, we construct G_1 as follows.

- For every clause C_j with literals u , v , and w , add the eight vertices c^j , d^j , u^j , v^j , w^j , $p^j(u)$, $p^j(v)$, and $p^j(w)$, add the four edges $c^j d^j$, $u^j p(u)^j$, $v^j p(v)^j$, and $w^j p(w)^j$, and let

$$E_j = \{c^j u^j, c^j v^j, c^j w^j\} \cup \{d^j p^j(u), d^j p^j(v), d^j p^j(w)\} \\ \cup \{u^j p^j(v), u^j p^j(w), v^j p^j(u), v^j p^j(w), w^j p^j(u), w^j p^j(v)\}.$$

See Figure 5 for an example.

- For every $i, j \in [m]$, and $k \in [n]$, add the edge $c^i d^j$, and, if the corresponding vertices exists, the edge $x_k^i \bar{x}_k^j$.

Let G_2 arise from G_1 by adding all edges in $E(\overline{G_1}) \setminus \bigcup_{j \in [m]} E_j$. Clearly, the size of (G_1, G_2) is polynomially bounded in terms of n and m .

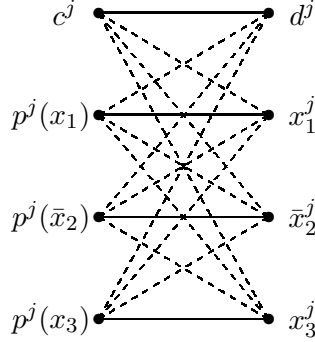


Figure 5: The vertices added for the clause $C_j = x_1 \vee \bar{x}_2 \vee x_3$. The dashed lines are the edges in E_j .

First, suppose that there is a truth assignment such that every clause of f contains exactly one true literal. Let

$$T = \bigcup_{j \in [m]} \{d^j\} \cup \bigcup_{i \in [n]: x_i \text{ is a true literal in } C_j} \{x_i^j\} \cup \bigcup_{i \in [n]: x_i \text{ is a false literal in } C_j} \{p(x_i)^j\}, \\ F = \bigcup_{j \in [m]} \{c^j\} \cup \bigcup_{i \in [n]: x_i \text{ is a true literal in } C_j} \{p(x_i)^j\} \cup \bigcup_{i \in [n]: x_i \text{ is a false literal in } C_j} \{x_i^j\}, \\ V(G) = V(G_1), \text{ and} \\ E(G) = \{e \in E(G_2) : |e \cap T| = 1\}.$$

Clearly, $G_1 \subseteq G \subseteq G_2$. Furthermore, if the clause C_j contains the three literals u , v , and w , and u is the true literal in C_j , then, within the graph G ,

- u^j is the only non-neighbor of c^j in T , c^j is the only non-neighbor of u^j in F ,
- $p(u)^j$ is the only non-neighbor of d^j in F , d^j is the only non-neighbor of $p(u)^j$ in T ,

- $p(w)^j$ is the only non-neighbor of v^j in T , v^j is the only non-neighbor of $p(w)^j$ in F ,
- $p(v)^j$ is the only non-neighbor of w^j in T , and w^j is the only non-neighbor of $p(v)^j$ in F .

Altogether, it follows that G solves the Π -SANDWICH PROBLEM.

Now, suppose that G solves the Π -SANDWICH PROBLEM. Let T and F denote a bipartition of G . In view of the edges of G_1 , we may assume that T contains the vertices d^1, \dots, d^m , and, that F contains the vertices c^1, \dots, c^m . Let $j \in [m]$, and let the clause C_j contain the literals u , v , and w . In view of the edges in E_j incident with c^j and d^j , respectively, it follows that T contains at most one vertex from $\{u, v, w\}$, and, that F contains at most one vertex from $\{p(u), p(v), p(w)\}$. Now, the edges in E_j between $\{u, v, w\}$ and $\{p(u), p(v), p(w)\}$ imply that T contains exactly one vertex from $\{u, v, w\}$. Hence, in view of the edges of the form $x_k^i \bar{x}_k^j$, setting the variables x_i that correspond to a vertex x_i^j in T to true yields a consistent truth assignment for which each clause of f contains exactly one true literal. \square

Theorem 3.8 *The $\{\text{paw}, \overline{\text{diamond}}\}$ -FREE SANDWICH DECISION PROBLEM is NP-complete.*

Proof: Let $\mathcal{F} = \{\text{paw}, \overline{\text{diamond}}\}$, and, let Π be as in Theorem 3.7. We describe a polynomial reduction of an instance (G_1, G_2) of the NP-complete Π -SANDWICH DECISION PROBLEM to an instance (G'_1, G'_2) of the \mathcal{F} -FREE SANDWICH DECISION PROBLEM. In view of the proof of Theorem 3.7, we may assume that G_1 has order at least 8, and contains no isolated vertex.

Let $P : a_1 b_1 a_2 b_2$ be an induced P_4 . Let G'_1 be the disjoint union of G_1 and P , and, let G'_2 arise from the disjoint union of G_2 and P by adding all edges between $V(G_2)$ and $V(P)$.

First, suppose that G solves the Π -SANDWICH PROBLEM, and, that the sets A and B form a suitable bipartition of G . If G' arises from the disjoint union of G and P by adding all edges between $\{a_1, a_2\}$ and B as well as all edges between $\{b_1, b_2\}$ and A , then $G' \in \mathcal{SW}_{\mathcal{F}}(G'_1, G'_2)$. Conversely, suppose that $\mathcal{SW}_{\mathcal{F}}(G'_1, G'_2)$ contains a graph G' . In view of P , some component of G' contains an induced $\overline{P_3}$. Since G' is $\overline{\text{diamond}}$ -free, this implies that G is connected. By Lemma 2.6, G' is $\{K_3, \overline{\text{diamond}}\}$ -free. Suppose that G' is not bipartite. Let $C : u_1 \dots u_\ell$ be a shortest odd cycle in G' . Since G' is triangle-free, ℓ is at least 5. Since G' is $\overline{\text{diamond}}$ -free, ℓ is at most 5, that is, ℓ is 5. Since G' has order more than 5, there is some vertex v in $V(G') \setminus V(C)$. Since G' is triangle-free, we may assume, by symmetry, that $N_G(v) \cap V(C)$ is contained in $\{u_1, u_3\}$. Now, $G'[\{u_2, u_4, u_5, v\}]$ is a $\overline{\text{diamond}}$, which is a contradiction. Hence, G' is bipartite. Let the sets A' and B' form a bipartition of G' with $a_1, a_2 \in A'$ and $b_1, b_2 \in B'$. Let $G = G' - V(P)$. Let $A = A' \setminus \{a_1, a_2\}$ and $B = B' \setminus \{b_1, b_2\}$. Suppose that some vertex a in A is non-adjacent to two vertices b and b' in B . Since G_1 has no isolated vertex, a has a neighbor b'' in B , and, $G[\{a, b, b', b''\}]$ is a $\overline{\text{diamond}}$, which is a contradiction. By symmetry, it follows that G solves the Π -SANDWICH PROBLEM \square

Theorem 3.9 *The $\{\text{diamond}, \overline{\text{diamond}}\}$ -FREE SANDWICH DECISION PROBLEM is NP-complete.*

Proof: Let $\mathcal{F} = \{\text{diamond}, \overline{\text{diamond}}\}$, and, let Π be as in Theorem 3.7. We describe a polynomial reduction of an instance (G_1, G_2) of the NP-complete Π -SANDWICH DECISION PROBLEM to an instance (G'_1, G'_2) of the \mathcal{F} -FREE SANDWICH DECISION PROBLEM. In view of the proof of Theorem 3.7, we may assume that G_1 has order at least 8, and contains no isolated vertex.

Let P' be the graph with vertices a_1, a_2, b_1, b'_1 , and b_2 , and edges $a_1 b_1, a_1 b'_1, b_1 a_2$, and $a_2 b_2$. Let G'_1 be the disjoint union of G_1 and P' , and, let G'_2 arise from the disjoint union of G_2 and P' by adding all edges between $V(G_2)$ and $V(P')$.

First, suppose that G solves the Π -SANDWICH PROBLEM, and, that the sets A and B form a suitable bipartition of G . If G' arises from the disjoint union of G and P' by adding all edges between $\{a_1, a_2\}$ and B as well as all edges between $\{b_1, b'_1, b_2\}$ and A , then $G' \in \mathcal{SW}_{\mathcal{F}}(G'_1, G'_2)$. Conversely, suppose that $\mathcal{SW}_{\mathcal{F}}(G'_1, G'_2)$ contains a graph G' . Since P contains an induced P_4 , some component of G' as well as some component of $\overline{G'}$ contains an induced $\overline{P_3}$. Since G' and $\overline{G'}$ are $\overline{\text{diamond}}$ -free, this implies that G and $\overline{G'}$ are connected. If G' is prime, then, since $\overline{G'}$ contains the triangle $b_1b'_1b_2$, a result of Brandstädt and Mahfud [4] implies that G' is a bipartite graph with partite sets A' and B' such that every vertex in A' has at most one non-neighbor in B' , and, every vertex in B' has at most one non-neighbor in A' . Now, $G = G' - V(P')$ solves the Π -SANDWICH PROBLEM. Hence, we may assume that G' contains a homogeneous set U of vertices, that is, $2 \leq |U| \leq n(G') - 1$, and $V(G') = U \cup A \cup B$, where A is the set of vertices in $V(G') \setminus U$ that are adjacent to every vertex in U , and, B is the set of vertices in $V(G') \setminus U$ that are adjacent to no vertex in U . Suppose that U contains two adjacent vertices. Since G' is diamond-free, A is a clique. Since G' is $\overline{\text{diamond}}$ -free, B is a clique. Since G' and $\overline{G'}$ are connected, we have $|A| \geq 2$ or $|B| \geq 2$. Since G' is \mathcal{F} -free, U is a clique. Nevertheless, this implies that $\overline{G'}$ is bipartite, which is impossible in view of the triangle $b_1b'_1b_2$ in $\overline{G'}$. Hence, we may assume that U contains two non-adjacent vertices. Arguing similarly as above, this implies that A , B , and U are independent, that is, G' is bipartite with bipartition A and $U \cup B$. If some vertex a in A has two non-neighbors b and b' in B , then a , b , b' , and a vertex from U induce a $\overline{\text{diamond}}$, which is a contradiction. If some vertex b in B has two non-neighbors a and a' in A , then, since G is connected, b has a neighbor a'' in A , and, b , a , a' , and a'' induce a $\overline{\text{diamond}}$, which is a contradiction. This completes the proof. \square

4 Conclusion

Figure 2 shows eight open cases, and, since the hardness results were typically slightly harder to obtain, we tend to believe that most of the corresponding problems are hard.

References

- [1] A. Brandstädt, $(P_5, \text{diamond})$ -free graphs revisited: structure and linear time optimization, Discrete Applied Mathematics 138 (2004) 13-27.
- [2] A. Brandstädt, J. Engelfriet, H.-O. Le, V. V. Lozin, Clique-width for 4-vertex forbidden subgraphs, Theory of Computing Systems 39 (2006) 561-590.
- [3] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph Classes: A Survey, Siam, 1999.
- [4] A. Brandstädt, S. Mahfud, Maximum Weight Stable Set on graphs without claw and co-claw (and similar graph classes) can be solved in linear time, Information Processing Letters 84 (2002) 251-259.
- [5] M. Cerioli, H. Everett, C.M.H. de Figueiredo, S. Klein, The homogeneous set sandwich problem, Information Processing Letters 67 (1998) 31-35.
- [6] K.K. Dabrowski, D. Paulusma, Clique-Width of Graph Classes Defined by Two Forbidden Induced Subgraphs, The Computer Journal 59 (2016) 650-666.
- [7] S. Dantas, C.M.H. de Figueiredo, M.V.G. da Silva, R.B. Teixeira, On the forbidden induced subgraph sandwich problem, Discrete Applied Mathematics 159 (2011) 1717-1725.

- [8] S. Dantas, C.M.H. de Figueiredo, F. Maffray, R.B. Teixeira, The complexity of forbidden subgraph sandwich problems and the skew partition sandwich problem, *Discrete Applied Mathematics* 182 (2015) 15-24.
- [9] S. Dantas, C.M.H. de Figueiredo, M.C. Golumbic, S. Klein, F. Maffray, The chain graph sandwich problem, *Annals of Operations Research* 188 (2011) 133-139.
- [10] S. Dantas, S. Klein, C.P. de Mello, A. Morgana, The Graph Sandwich Problem for P_4 -sparse graphs, *Discrete Mathematics* 309 (2009) 3664-3673.
- [11] M.R. Garey, D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman & Co., 1979.
- [12] M.C. Golumbic, H. Kaplan, R. Shamir, Graph sandwich problems, *Journal of Algorithms* 19 (1995) 449-473.
- [13] M.C. Golumbic, Matrix sandwich problems, *Linear Algebra and Applications* 277 (1998) 239-251.
- [14] N. Korpelainen, V. Lozin, Two forbidden induced subgraphs and well-quasi-ordering, *Discrete Mathematics* 311 (2011) 1813-1822.
- [15] F. Maffray, M. Preissmann, Linear recognition of pseudo-split graphs, *Discrete Applied Mathematics* 52 (1994) 307-312.
- [16] N.V.R. Mahadev, U.N. Peled, *Threshold Graphs and Related Topics*, *Annals of Discrete Mathematics* 56, Elsevier, 1995.
- [17] S. Olariu, Paw-free graphs, *Information Processing Letters* 28 (1988) 53-54.
- [18] R.B. Teixeira, S. Dantas, C.M.H. de Figueiredo, The polynomial dichotomy for three nonempty part sandwich problems, *Discrete Applied Mathematics* 158 (2010) 1286-1304.
- [19] R.B. Teixeira, S. Dantas, C.M.H. de Figueiredo, The external constraint 4 nonempty part sandwich problem, *Discrete Applied Mathematics* 159 (2011) 661-673.